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Chain Recurrent Subsets of ∂R_+^p as ω-Limit Sets

by

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I. Preliminaries and results

Let (E, d) be a locally compact metric space and let $\pi: R\times E\to E$ be a dynamical system. Consider an $x\in E$ and assume that the positive semitrajectory through x has compact closure. Then $\omega(x)$, the ω-limit set of x is a nonempty, compact, connected invariant subset of E [3] and $\pi|\omega(x)$, the restriction of π onto $\omega(x)$ is chain recurrent [11, Thm. II. 6.3.C.]. Since we use standard notation and terminology [3], [11], we restrict ourselves here to recall only the definition of chain recurrence. A nonempty compact invariant subset Y of E is said to be *chain recurrent* [11] if for all $\varepsilon>0$, $T>0$, $x, y\in Y$ there are finite sequences $\{x_n\}_0^{N+1}\subset Y$, $\{t_n\}_0^N\subset[T, \infty)$ such that $x_0=x$, $x_{N+1}=y$ and $d(\pi(t_n, x_n), x_{n+1})<\varepsilon$, $n=0, 1, 2, \cdots, N$.

A converse statement was proved by R. Bowen [5]: if (W, d) is a compact metric space, $\rho: R\times W\to W$ is a dynamical system and W is chain recurrent, then there exist a compact metric space (Z, Δ) and a dynamical system $\xi: R\times Z\to Z$ such that $W\subset Z$, $\Delta\,|\,W=d$, $\xi\,|\,W=\rho$ and for all $z\in Z\setminus W$, there holds $\omega(z)=W$. In other words, chain recurrent sets are ω-limit sets for suitable extensions of the dynamical system and such extensions can be defined by adjoining a single new trajectory. The original result in [5] was formulated and proved for discrete dynamical systems but (as it is observed in [11, p. 38]; with essentially the same proof) it holds equally true for dynamical systems with continuous time.

Let X be a closed subset of E and assume that ∂X, the boundary of X in E is invariant. In recent years, motivated by persistence theory, a part of mathematical biology [23], several papers were devoted to study the behaviour of $\pi\,|\,X$ in a small vicinity of ∂X. In almost all of these investigations, subsets of ∂X which are ω-limit sets of points lying in $X\setminus\partial X$ have played an important role [7], [8], [9], [15], [17], [18], [22], [23]. Some of the results were generalized for dynamical systems on abstract metric spaces with the compact attraction property [19]. In most applications [23], $E=R^p$, $X=R_+^p$, the non-negative orthant in R^p, and π is induced by a system of ecological differential equations. It is usually assumed that π is ecologically stable, i.e.

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$$\sup\{\limsup\{\|\pi(t, x)\| \mid t\to\infty\} \mid x\in R^p_+\}<\infty .$$

The state variable $x=(x^1, x^2, \cdots, x^p)\in R^p_+$ is interpreted as the frequency vector of the different species of a given ecosystem. In this context, the existence of a point $x_0\in R^p_+\setminus\partial R^p_+$ for which $\omega(x_0)\subset\partial R^p_+$ is understood as a strong violation of persistence (the ultimate survival of all species).

A HISTORICAL REMARK. A central tool in persistence theory is the application of the so-called Butler-McGehee Lemma [17, Lemma A.1]. In the early eighties, H. I. Freedman and P. Waltman, in collaboration with G. J. Butler and R. McGehee, have found a powerful method for detecting subsets of ∂R^p_+ which are ω-limit sets of points lying in $R^p_+\setminus\partial R^p_+$. Very cautiously, admitting that their method might have been known earlier in abstract topological dynamics, they formulated it as a separate Lemma in the Appendix of their paper. In 1989, treating persistence theory within the general framework of Conley's theory [11] on isolated invariant sets, both J. Hofbauer [23] and the present author [18] observed that the Butler-McGehee Lemma is a special case of a much earlier result due to T. Ura and I. Kimura [28] (for a restatement and a short discussion of the original result in [28], see [3, Chapter VI] esp. [3, Cor VI. 1.2]). A discrete version (in abstract metric spaces, for semidynamical systems with the compact attraction property, an abstract counterpart of ecological stability) of the Ura-Kimura result was proved in [14]. Moreover, though using completely different terminology, C. C. Fenske and H. O. Peitgen have interpreted one of their results as an abstract theorem on persistence [14, Lemma 2.4.]. To the best of our knowledge, the geometrical idea underlying the Butler-McGehee Lemma has appeared firstly in a paper by W. H. Gottschalk [19]. In arbitrary locally compact noncompact metric spaces G, he proved the nonexistence of dynamical systems $\rho: R\times G\to G$ with the property that $\omega(x)=G$ for all $x\in G$. (Here is a short proof: by letting $\hat{\rho}(t, x)=\rho(t, x)$ whenever $t\in R$; $x\in G$ and Ω whenever $t\in R$, $x=\Omega$, we define a dynamical system on the one-point-compactification $\hat{G}$ of G. (Here, of course, Ω is the point at infinity.) Applying [3, Cor VI. 1.2] for Ω, we arrive at an immediate contradiction.) It is worth to mention here that the related question whether R^p, $p\geqslant 3$ can be a minimal set for some dynamical systems, is still unsolved [13]. Finally, we would call the attention to a forthcoming paper of H. I. Freedman and P. Moson [16] where different persistence definitions are treated in the light of the duality theory between boundedness and stability concepts of topological dynamics established by J. Auslander and P. Seibert [2].

The aim of this paper is to study a question somewhat complementary to the development outlined in the previous remark. Instead of starting from a dynamical system $\pi: R\times X\to X$ and finding conditions guaranteeing or excluding the existence of a point $x_0\in X\setminus\partial X$ with $\omega(x_0)\subset\partial X$, we start from a dynamical system ρ defined on ∂X and from a chain recurrent subset M of ∂X and our aim is to extend ρ to X so that $M=\omega(x_0)$ for some $x_0\in X\setminus\partial X$.

There is no hope to solve this problem in full generality. The reason is that (observe that for all $t \in R$, the mapping $\partial X \to \partial X$, $x \to \rho(t, x)$ is a homeomorphism) all extension results for homeomorphisms are operating with very special conditions. A short list of classical results should probably contain Schönfliess theorem [26], Hausdorff's extension theorm for metrics [29], Thom's isotopy extension theorem [6, Chapter 9] and certain results (see e.g. in [12]) of infinite-dimensional topology. On the other hand, the extension problem for continuous functions is satisfactorily solved (Tietze theorem, Dugundji theorem and various results on absolute neighborhood retracts and extensors, see e.g. [4]). It is one of these classical results, namely a simple formula attributed to S. Banach in [10] for extending globally Lipschitzian functions (i.e. functions satisfying a global Lipschitz condition), which enables us to prove the following:

THEOREM. *Let $f: R^{p-1} \to R^{p-1}$ be a bounded globally Lipschitzian function. The conditions imposed on f imply that the solutions of the ordinary differential equation $d_t x = f(x)$, $x \in R^{p-1}$ define a dynamical system on R^{p-1}. Assume that M is a chain recurrent subset of R^{p-1}. Then there is a bounded continuous function $F: R^{p-1} \times R_+ \to R^p$ with the properties that $F(x, 0) = (f(x), 0)$ whenever $x \in R^{p-1}$, the solutions of the ordinary differential equation $d_t(x, \lambda) = F(x, \lambda)$, $(x, \lambda) \in R^{p-1} \times R_+$ define a dynamical system on $R^{p-1} \times R_+$ and there is an $(x_0, \lambda_0) \in R^{p-1} \times (R_+ \setminus \{0\})$ with $\omega((x_0, \lambda_0)) = \{(m, 0) \in R^{p-1} \times R_+ \mid m \in M\}$.*

The proof of the theorem is postponed to Section 2. See also the technical remark at the end of the proof.

COROLLARY. *Let $\rho: R \times \partial R^p_+ \to \partial R^p_+$ be a dynamical system. Assume that ρ is induced by the solutions of the ordinary differential equation $d_t x = g(x)$, $x \in \partial R^p_+$ where $g: \partial R^p_+ \to R^p$ is a bounded globally Lipschitzian function. Further, assume that M is a chain recurrent subset of ∂R^p_+. Then there is a bounded continuous function $G: R^p_+ \to R^p$ with the properties that $G \mid \partial R^p_+ = g$, the solutions of the ordinary differential equation $d_t x = G(x)$, $x \in R^p_+$ define a dynamical system on R^p_+ and there is an $x_0 \in R^p_+ \setminus \partial R^p_+$ with $\omega(x_0) = M$.*

Proof. In virtue of the straightening-the-angle-procedure [6, p. 146–148] i.e. by constructing a homeomorphism of the pair $(R^p_+, \partial R^p_+)$ onto the pair $(R^{p-1} \times R_+, R^{p-1} \times \{0\})$ with the properties that both h and h^{-1} are globally Lipschitzian and that for each subset $S \subset \{1, 2, \cdots, p\}$, the restriction of h onto the invariant subset $\{(x^1, x^2, \cdots, x^p) \in R^p_+ \mid x^i = 0 \text{ if } i \in S \text{ and } x^i > 0 \text{ if } i \notin S\}$ is a C^∞ diffeomorphism, the desired result follows from the transformation formulae for ordinary differential equations.

For an other extension result in biomathematics, see the construction in [27, Example 3] pointing out that ecological differential equations commonly used to describe competiting species are compatible with any dynamical behaviour.

II. The proof of the Theorem

Besides the extension formula for globally Lipschitzian functions, the proof makes use of only elementary approximation and ordinary differential equations techniques. For example we use a well-known global version of the Picard-Lindelöf theorem (see e.g. [25, Thm. 2.8.1]) implying that the solutions of the differential equation $d_t x = f(x)$ (as well as those of $d_t z = f_{n+1}(z)$, $n \in N$) define a dynamical system. We make use of the basic results on dependence of solutions upon initial conditions [21, Chapter V] as well. Both in R^{p-1} and R^p, the Euclidean norm will be denoted by $\|\cdot\|$. The induced matrix norm is also denoted by $\|\cdot\|$. Though double bars are denoting norms in different spaces, no confusion should arise.

The proof is subdivided into several steps.

A preparatory step: The dynamical system defined by the solutions of $d_t x = f(x)$, $x \in R^{p-1}$ is denoted by π. Argueing as in [5, p. 334], we can find sequences $\{a_n\} \subset M$, $\{b_n\} \subset M$, $\{t_n\} \subset R^+$ with the properties that

for each $n \in N$, the set $\{a_n, a_{n+1}, \cdots\}$ is dense in M, for each $n \in N$, there holds $\pi(t_n, a_n) = b_n$ and $7 \leqslant t_n \leqslant 8$ and $\|b_n - a_{n+1}\| \to 0$ as $n \to \infty$.

The first part of the construction: Choose $K, L \in R_+$ so that $\|f(x)\| \leqslant K$ and $\|f(x) - f(z)\| \leqslant L\|x - z\|$ for all $x, z \in R^{p-1}$. Consider a C^∞ function $H\colon R^{p-1} \to R_+$ for which

$$H(w) = 0 \quad \text{whenever} \quad \|w\| \geqslant 1 \quad \text{and} \quad \int_{R^{p-1}} H(w)dw = 1\,.$$

For $n \in N$, define

$$f_n(x) = (n+1)^{p-1} \int_{R^{p-1}} f(w+x)H((n+1)w)dw\,, \qquad x \in R^{p-1}\,.$$

It is elementary to check that $f_n\colon R^{p-1} \to R^{p-1}$ is a C^∞ function and

$$\|f_n(x) - f_n(z)\| \leqslant L\|x - z\| \qquad \text{whenever} \quad x, z \in R^{p-1}\,,$$

$$\|f_n(x) - f(x)\| \leqslant L/(n+1) \quad \text{and} \quad \|f_n(x)\| \leq K \quad \text{for all} \quad x \in R^{p-1}\,.$$

Starting with $T_0 = 0$, define inductively $T_{n+1} = T_n + t_n$, $n \in N$ and choose a C^∞ increasing function $\mu_n\colon R \to R_+$ such that $\mu_n(t) = 0$ whenever $t \leqslant T_n + 1$ and 1 whenever $t \geqslant T_n + 2$. We may assume that $\dot{\mu}_n(t) \leqslant 2$ for all $t \in R$. For $n \in N$, define

$$g_n(x, t) = \mu_n(t) f_{n+1}(x) + (1 - \mu_n(t)) f_n(x)\,, \qquad x \in R^{p-1}\,, \quad t \in R\,.$$

It is easy to see that $g_n\colon R^{p-1} \times R \to R^{p-1}$ is a C^∞ function and

$$\|g_n(x, t) - g_n(z, t)\| \leqslant L\|x - z\| \qquad \text{whenever} \quad x, z \in R^{p-1}\,, \quad t \in R\,,$$

$$\|g_n(x, t) - f(x)\| \leqslant L/(n+1) \qquad \text{and} \quad \|g_n(x, t)\| \leqslant K \quad \text{for all} \quad x \in R^{p-1},\ t \in R\,.$$

The solution of the initial value problem (we obviously have existence, uniqueness and that the unique nonextendable solution is defined for all $t \in R$)

$$d_t x = g_n(x, t), \qquad x(T_n) = a_n$$

is denoted by x_n. For brevity, we write $c_n = x_n(T_n + 3)$, $n \in N$. Let $\Phi_{n+1}: R \times R \times R^{p-1} \to R^{p-1}$ be the solution operator to the ordinary differential equation

$$d_t z = f_{n+1}(z), \qquad z \in R^{p-1}.$$

Then $z_{n+1}: R \to R^{p-1}$ defined by $z_{n+1}(t) = \Phi_{n+1}(t, T_{n+1}, a_{n+1})$ is the solution satisfying the initial conditions $z(T_{n+1}) = a_{n+1}$. For brevity, we write $e_n = z_{n+1}(T_n + 3)$, $n \in N$.

For $n \in N$, choose a C^∞ increasing function $\kappa_n: R \to R_+$ such that $\kappa_n(t) = 0$ whenever $t \leqslant T_n + 4$ and 1 whenever $t \geqslant T_{n+1} - 1$. We may assume that $\dot{\kappa}_n(t) \leqslant 1$ for all $t \in R$. Now we are in a position to define an important auxiliary function $q: R \to R^{p-1}$ (to be interpreted later as the projection of a trajectory of the extended dynamical system onto R^{p-1}). For $t \in R$, let

$$q(t) = \begin{cases} x_0(t) & \text{if } \; t \leqslant 1 = T_0 + 1 \\ x_n(t) & \text{if } \; t \in [T_n - 1, T_n + 4] \\ \Phi_{n+1}(t, T_n + 3, c_n + \kappa_n(t)(e_n - c_n)) & \text{if } \; t \in [T_n + 2, T_{n+1} + 1]. \end{cases}$$

Observe that $q(t)$ is well-defined. In fact, we have that $g_n(x, t) = f_n(x)$ whenever $t \leqslant T_n + 1$ and $f_{n+1}(x)$ whenever $t \geqslant T_n + 2$. Similarly, we have that $c_n + \kappa_n(t)(e_n - c_n) = c_n$ whenever $t \leqslant T_n + 4$ and e_n whenever $t \geqslant T_{n+1} - 1$. It follows immediately that

$$x_n(t) = \Phi_n(t, T_{n-1} + 3, e_{n-1}) \qquad \text{for all} \quad t \in [T_n - 1, T_n + 1], \quad n \in N \setminus \{0\}$$

and

$$x_n(t) = \Phi_{n+1}(t, T_n + 3, c_n) \qquad \text{for all} \quad t \in [T_n + 2, T_n + 4], \quad n \in N.$$

In virtue of the theorem on the differentiable dependence of the solutions upon initial conditions, the previous consideration shows also that the function $q: R \to R^{p-1}$ is of class C^∞.

Some computational Lemmas: We state and prove some important properties of the function q.

LEMMA 1. *There holds* $\|c_n - e_n\| \to 0$ *as* $n \to \infty$.

Proof. Let $y_n: R \to R^{p-1}$ denote the solution of the initial value problem

$$d_t y = f(t), \qquad y(T_n) = a_n.$$

Observe that $y_n(T_{n+1}) = b_{n+1}$ and for brevity, we write $d_n = y_n(T_n + 3)$, $n \in N$.

For all $t \geqslant T_n$, since

$$x_n(t) = a_n + \int_{T_n}^t g_n(x_n(s), s) ds \quad \text{and} \quad y_n(t) = a_n + \int_{T_n}^t f(y_n(s)) ds$$

we have that

$$x_n(t)-y_n(t)=\int_{T_n}^{t}(g_n(x_n(s), s)-f(x_n(s)))ds+\int_{T_n}^{t}(f(x_n(s))-f(y_n(s)))ds\,,$$

$$\|x_n(t)-y_n(t)\|\leq\int_{T_n}^{t}(L/(n+1))ds+\int_{T_n}^{t}L\|x_n(s)-y_n(s)\|ds$$

and therefore, applying Gronwall's lemma with $t=T_n+3$,

$$\|c_n-d_n\|\leqslant 3L(n+1)^{-1}\exp(3L)\,,\qquad n\in N\,.$$

Similarily, starting from

$$y_n(t)=b_n+\int_{T_{n+1}}^{t}f(y_n(s))ds \quad\text{and}\quad z_{n+1}(t)=a_{n+1}+\int_{T_{n+1}}^{t}f_{n+1}(z_{n+1}(s))ds$$

and using $T_{n+1}-(T_n+3)=t_n-3$, we obtain that

$$\begin{aligned}y_n(t)-z_{n+1}(t)&=b_n-a_{n+1}\\&\quad+\int_{T_{n+1}}^{t}(f(y_n(s))-f_{n+1}(y_n(s)))ds+\int_{T_{n+1}}^{t}(f_{n+1}(y_n(s))-f_{n+1}(z_{n+1}(s)))ds\,,\end{aligned}$$

$$\|d_n-e_n\|\leqslant(\|b_n-a_{n+1}\|+(t_n-3)L(n+2)^{-1})\exp((t_n-3)L)\,,\qquad n\in N\,.$$

Recall that $t_n-3\in[4, 5]$ for all $n\in N$. Consequently,

$$\|c_n-e_n\|\leqslant\|c_n-d_n\|+\|d_n-e_n\|\to 0\qquad\text{as}\quad n\to\infty\,.$$

LEMMA 2. *For each $n\in N$, there holds*

$$\|d_tx_n(s)-d_tx_n(u)\|\leqslant K(L+4)|s-u|\qquad\textit{whenever}\quad s, u\in R\,.$$

Proof: Using the basic properties of g_n, f_n and μ_n, we have that

$$\begin{aligned}\|d_tx_n(s)-d_tx_n(u)\|&=\|g_n(x_n(s), s)-g_n(x_n(u), u)\|\\&\leqslant\|g_n(x_n(s), s)-g_n(x_n(u), s)\|+\|g_n(x_n(u), s)-g_n(x_n(u), u)\|\\&\leqslant L\|x_n(s)-x_n(u)\|+\|(f_{n+1}(x_n(u))-f_n(x_n(u)))\cdot(\mu_n(s)-\mu_n(u))\|\\&\leqslant L\left\|\int_u^s g_n(x_n(t), t)dt\right\|+2K|\mu_n(s)-\mu_n(u)|\leqslant LK|s-u|+4K|s-u|\,.\end{aligned}$$

LEMMA 3. *For each $k\in N$, there exists a $P_k\in R$ such that*

$$\|d_tq(s)-d_tq(u)\|\leqslant P_k|s-u|\qquad\textit{whenever}\quad s, u\leqslant T_k\,.$$

Proof. For each $n\in N$, we point out the existence of a $Q_n\in R$ with the property that for all $s, u\in[T_n+2, T_{n+1}+1]$, there holds

$$\begin{aligned}\|d_tq(s)-d_tq(u)\| = \|&d_t\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n))\\ &+d_x\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n))\dot\kappa_n(s)(e_n-c_n)\\ &-d_t\Phi_{n+1}(u, T_n+3, c_n+\kappa_n(u)(e_n-c_n))\\ &-d_x\Phi_{n+1}(u, T_n+3, c_n+\kappa_n(u)(e_n-c_n))\dot\kappa_n(u)(e_n-c_n)\| \leqslant Q_n|s-u|\,.\end{aligned}$$

Here of course, $d_t\Phi_{n+1}$ resp. $d_x\Phi_{n+1}$ denote the partial derivative of Φ_{n+1} with respect to the first resp. third variable. In virtue of the theorem on the differentiable dependence of the solutions upon initial conditions, all partial derivatives of Φ_{n+1} exist. The desired inequality follows from the simple observation that continuously differentiable functions (namely the partial derivatives of Φ_{n+1}) when restricted to compact sets are globally Lipschitzian and that sums, products and compositions of Lipschitzian functions are also Lipschitzian. In virtue of the previous Lemma, it is obvious that P_k can be chosen for $P_k=\max\{K(L+4), Q_0, Q_1, \cdots, Q_{k-1}\}$.

LEMMA 4. *There holds* $\|d_tq(s)-f(q(s))\|\to 0$ *as* $s\to\infty$.

Proof. We begin by observing that

$$\|d_tx_n(s)-f(x_n(s))\| = \|g_n(x_n(s), s)-f(x_n(s))\| \leqslant L/(n+1)$$

for all $s\in R$, $n\in N$.

Using the basic properties of f_{n+1} and κ_n, we have that

$$\begin{aligned}\|d_tq(s)-f(q(s))\| &\leqslant \|d_x\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n))\dot\kappa_n(s)(e_n-c_n)\|\\ &\quad+\|d_t\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n))-f(\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n)))\|\\ &=\|d_x\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n))\dot\kappa_n(s)(e_n-c_n)\|\\ &\quad+\|f_{n+1}(\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n)))-f(\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n)))\|\\ &\leqslant \|d_x\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n))\|\cdot\|e_n-c_n\|+L/(n+2)\end{aligned}$$

for all $s\in[T_n+2, T_{n+1}+1]$, $n\in N$. Thus, in virtue of Lemma 1, it is sufficient to point out the existence of a constant Q with

$$\|d_x\Phi_{n+1}(s, T_n+3, c_n+\kappa_n(s)(e_n-c_n))\|\leqslant Q$$

whenever $s\in[T_n+2, T_{n+1}+1]$, $n\in N$.

Fix $n\in N$ and $s\in[T_n+2, T_{n+1}+1]$ arbitrarily. For brevity, we write

$$w(u)=\Phi_{n+1}(u, T_n+3, c_n+\kappa_n(s)(e_n-c_n))\,,\qquad u\in R\,,$$

$$W(u)=d_x\Phi_{n+1}(u, T_n+3, c_n+\kappa_n(s)(e_n-c_n))\,,\qquad u\in R\,.$$

It is well-known that the matrix-valued function W satisfies the variational equation $d_uW=[d_xf_{n+1}(w(u))]W$ and $W(T_n+3)=I$, the unit matrix. Recall that f_{n+1} is of class C^∞ and $\|f_{n+1}(x)-f_{n+1}(z)\|\leqslant L\|x-z\|$ whenever $x, z\in R^{p-1}$. Consequently, $\|d_xf_{n+1}(y)\|\leqslant L$ for all $y\in R^{p-1}$ and a simple application of Gronwall's lemma yields that $\|W(u)\| \leqslant \exp(L|u-(T_n+3)|)$, $u \in R$. In particular, Q can be chosen for $\exp(6 \cdot L)$.

The second part of the construction: Choose a strictly decreasing convex C^∞ function $v: R \to R_+$ with the properties that $v(t)=1-t$ whenever $t \leqslant 0$ and $v(t) \to 0$ as $t \to \infty$. Set

$$A=\{(x, v(t)) \in R^{p-1} \times R_+ \subset R^p \mid t \leqslant 0\},$$
$$B_n=\{(x, v(t)) \in R^{p-1} \times R_+ \subset R^p \mid t=T_n\}, \qquad n \in N,$$
$$C=\{(q(t), v(t)) \in R^{p-1} \times R_+ \subset R^p \mid t \in R\},$$
$$S=A \cup \{B_n \mid n \in N\} \cup C.$$

By letting

$$\psi(x, \lambda)=\begin{cases} f_0(x) & \text{if } (x, \lambda) \in A \\ f_n(x) & \text{if } (x, \lambda) \in B_n \text{ for some } n \in N \\ d_t q(s) & \text{if } (x, \lambda)=(q(s), v(s)) \in C \text{ for some } s \in R, \end{cases}$$

we define a function $\psi: S \to R^{p-1}$. Observe that ψ is well-defined. In fact, by definition, $q(s)=x_n(s)$ and therefore, $d_t q(s)=d_t x_n(s)=g_n(x_n(s), s)=f_n(x_n(s))=f_n(q(s))$ whenever $s \in [T_n-1, T_n+1]$, $n \in N$. Similarly, $d_t q(s)=f_0(q(s))$ for all $s \leqslant 0$.

Define

$$W_n=S \cap \{(x, \lambda) \in R^{p-1} \times R_+ \subset R^p \mid v(T_{n+1}) \leqslant \lambda \leqslant v(T_n)\}, \qquad n \in N.$$

LEMMA 5. *For each $n \in N$, there is a constant H_n such that*

$$\|\psi(x, \lambda)-\psi(y, \mu)\| \leqslant H_n \|(x, \lambda)-(y, \mu)\| \qquad \text{for all} \quad (x, \lambda), (y, \mu) \in W_n.$$

Proof. Observe that $W_n=(C \cap W_n) \cup B_n \cup B_{n+1}$. Hence we have to distinguish three cases according as $(x, \lambda) \in C \cap W_n$, $(x, \lambda) \in B_n$ or $(x, \lambda) \in B_{n+1}$. We restrict ourselves to the first case: the rest can be treated similarly. We may represent (x, λ) in the form $(x, \lambda)=(q(s), v(s))$ for some $s \in [T_n, T_{n+1}]$. There are three cases again according as $(y, \mu) \in C \cap W_n$, $(y, \mu) \in B_n$ or $(y, \mu) \in B_{n+1}$.

Assume first that $(y, \mu) \in C \cap W_n$ i.e. $(y, \mu)=(q(u), v(u))$ for some $u \in [T_n, T_{n+1}]$. Since $\|(x, \lambda)-(y, \mu)\| \geqslant |\lambda-\mu|=|v(s)-v(u)| \geqslant -\dot{v}(\max\{s, u\})|s-u|$, an application of Lemma 3 yields that

$$\|\psi(x, \lambda)-\psi(y, \mu)\|=\|d_t q(s)-d_t q(u)\| \leqslant P_{n+1}|s-u| \leqslant (-P_{n+1}/\dot{v}(T_{n+1}))\|(x, \lambda)-(y, \mu)\|.$$

Secondly, assume that $(y, \mu) \in B_n$. Since $q: R \to R^{p-1}$ is continuously differentiable, there is a constant J_n such that $\|d_t q(s)\| \leqslant J_n$ for all $s \in [T_n, T_{n+1}]$. Therefore, if $s \in [T_n+1, T_{n+1}]$, then

$$\|\psi(x, \lambda)-\psi(y, \mu)\|=\|d_t q(s)-f_n(y)\| \leqslant \|d_t q(s)\|+\|f_n(y)\| \leqslant J_n+K$$
$$=((J_n+K)/(v(T_n)-\lambda))|\lambda-\mu| \leqslant ((J_n+K)/(v(T_n)-v(T_n+1)))\|(x, \lambda)-(y, \mu)\|.$$

On the other hand, if $s \in [T_n, T_n+1]$, then

$$\|\psi(x,\lambda)-\psi(y,\mu)\| = \|f_n(x)-f_n(y)\| \leqslant L\|x-y\| \leqslant L\|(x,\lambda)-(y,\mu)\| .$$

The third case i.e. when $(y,\mu)\in B_{n+1}$ can be settled similarly. The constant H_n can be chosen for the greatest Lipschitz constant computed in the separate cases.

LEMMA 6. *There is a sequence* $\{\delta_n\}\subset R_+$ *such that* $\|\psi(x,\lambda)-f(x)\|\leqslant\delta_n$ *whenever* $(x,\lambda)\in W_n$, $n\in N$ *and* $\delta_n\to 0$ *as* $n\to\infty$.

Proof. This is a simple consequence of Lemma 4 and of the inequalities $\|f_n(x)-f(x)\|\leqslant L/(n+1)$, $x\in R^{p-1}$, $n\in N$.

The co-ordinate functions of f_0, f and ψ are denoted by $f_0^1, f_0^2,\cdots,f_0^{p-1}$, $f^1,f^2,\cdots,f^{p-1}$ and $\psi^1,\psi^2,\cdots,\psi^{p-1}$, respectively. Define

$$Z_n=\{(x,\lambda)\in R^{p-1}\times R_+\subset R^p \mid v(T_{n+1})\leqslant\lambda\leqslant v(T_n)\}, \qquad n\in N .$$

Observe that $W_n=S\cap Z_n$. For $k=1,2,\cdots,p-1$, Lemma 5 implies that $\psi^k\mid W_n$: $W_n\to R$ is Lipschitzian with Lipschitz constant H_n. For $(x,\lambda)\in Z_n$, set

$$\varphi_n^k(x,\lambda)=\inf\{\psi^k(y,\mu)+H_n\|(x,\lambda)-(y,\mu)\| \mid (y,\mu)\in W_n\} ,$$
$$F_n^k(x,\lambda)=\max\{f^k(x)-\delta_n,\ \min\{\varphi_n^k(x,\lambda),\ f^k(x)+\delta_n\}\} .$$

By the extension theorem in [10], φ_n^k: $Z_n\to R$ is Lipschitzian with Lipschitz constant H_n and $\varphi_n^k\mid W_n=\psi^k\mid W_n$. It is easy to see that $|F_n^k(x,\lambda)-f^k(x)|\leqslant\delta_n$ and that F_n^k: $Z_n\to R$ is Lipschitzian with Lipschitz constant $\max\{H_n, L\}$. Since $f^k(x)-\delta_n\leqslant\psi^k(x,\lambda)\leqslant f^k(x)+\delta_n$ for all $(x,\lambda)\in W_n$ (this is a corollary of Lemma 6), there holds $F_n^k\mid W_n=\varphi_n^k\mid W_n=\psi^k\mid W_n$. Applying Lemma 6 again, we conclude that

$$F^k(x,\lambda)=\begin{cases} f_0^k & \text{if } (x,\lambda)\in A \\ F_n^k(x,\lambda) & \text{if } (x,\lambda)\in Z_n,\ n\in N \\ f^k(x) & \text{if } (x,\lambda)\in R^{p-1}\times\{0\} \end{cases}$$

defines a continuous function F^k: $R^{p-1}\times R_+\to R$, $k=1,2,\cdots,p-1$. By the construction, $F^k\mid R^{p-1}\times(R_+\setminus\{0\})$ is bounded and locally Lipschitzian.

For each $\lambda>0$, equation $v(t)=\lambda$ has a unique solution for t. Consequently, definition

$$F^p(x,\lambda)=\begin{cases} \dot v(t) & \text{if } (x,\lambda)\in R^{p-1}\times(R_+\setminus\{0\}),\ v(t)=\lambda \\ 0 & \text{if } (x,\lambda)\in R^{p-1}\times\{0\} \end{cases}$$

makes sense. By the properties of v, the function F^p: $R^{p-1}\times R_+\to R$ is continuous and $F^p\mid R^{p-1}\times(R_+\setminus\{0\})$ is bounded and locally Lipschitzian.

Consider now the function F: $R^{p-1}\times R_+\to R^p$ defined by

$$F(x,\lambda)=(F^1(x,\lambda), F^2(x,\lambda),\cdots,F^p(x,\lambda)), \qquad (x,\lambda)\in R^{p-1}\times R_+ .$$

Observe that $F(x,0)=(f(x),0)$ for all $x\in R^{p-1}$ and $F\mid R^{p-1}\times(R_+\setminus\{0\})$ is bounded and locally Lipischitzian. For $(x^0,\lambda^0)\in R^{p-1}\times R_+$ arbitrary, consider the initial value problem

$$d_t(x, \lambda)=F(x, \lambda), \qquad (x(0), \lambda(0))=(x^0, \lambda^0).$$

The differential equation for λ does not depend on x. Moreover, it can be solved explicitly. If $\lambda^0>0$, the unique solution $\Lambda^0: R\to R^+$ is given by $\Lambda_0(t)=v(t+\tau^0)$ where $\tau^0\in R$ is determined by $\lambda^0=v(\tau^0)$. If $\lambda^0=0$, the unique solution $\Lambda_0: R\to R^+$ is identically zero. Substituting $\Lambda^0(t)$ for λ into the first $p-1$ equations, we obtain the initial value problem $d_tx=F(x, \Lambda^0(t))$, $x(0)=x^0$. The right-hand side is bounded and locally Lipschitzian in x. Therefore, it has a unique solution which is defined for all $t\in R$. Since uniqueness implies continuous dependence upon initial conditions, it follows that the solutions of the ordinary differential equation $d_t(x, \lambda)=F(x, \lambda)$, $(x, \lambda)\in R^{p-1}\times R_+$ define a dynamical system on $R^{p-1}\times R_+$.

For $s\in R$ arbitrary, consider now the point $(q(s), v(s))\in C$. By the construction, there holds

$$\begin{aligned}F(q(s), v(s))&=(F^1(q(s), v(s)), \cdots, F^{p-1}(q(s), v(s)), F^p(q(s), v(s)))\\&=(\psi(q(s), v(s)), \dot v(s))=(d_tq(s), \dot v(s)).\end{aligned}$$

In other words, $x=q(s)$, $\lambda=v(s)$ is the solution to the initial value problem $d_t(x, \lambda)=F(x, \lambda)$, $x(0)=q(0)=a_0$, $\lambda(0)=v(0)=1$. Recall that $q(T_n)=a_n$, $n\in N$. Consequently, since $v(t)\to 0$ as $t\to\infty$ and $\{a_n, a_{n+1}, \cdots\}$ is dense in M for all $n\in N$, we have that $\omega((a_0, 1))=\{(m, 0)\in R^{p-1}\times R_+ \mid m\in M\}$.

The proof of the Theorem is completed.

A TECHNICAL REMARK. Unfortunately, the function F constructed above is not necessarily Lipschitzian at points of $R^{p-1}\times\{0\}$. The difficulty is that we are not able to ensure that $\psi \mid C\cup(R^{p-1}\times\{0\})$ be Lipschitzian. However, it is worth to mention here that a ramification of our method (a better choice for v plus sharper estimates: for example, in proving our Theorem it was enough to refer to a qualitative result in the proof of Lemma 3—explicit estimates give that P_k does not depend on k provided that $|\ddot\kappa_n(t)|\leqslant C$ for some constant C (independent of t and n)) yields that $\psi \mid C\cup(R^{p-1}\times\{0\})$ is globally Lipschitzian provided that $\sum\|b_n-a_{n+1}\|<\infty$, a rather unpleasant technical condition which can not be taken for granted. This difficulty can not be overcome by assuming that f is of class C^1 or C^∞. On the other hand, the proof of the Theorem can easily be modified so as to remain valid in the case when the conditions imposed on f are weakened to the requirement that f is globally Lipschitzian in a neighborhood of M.

To ensure better properties of F in neighborhoods of points of $R^{p-1}\times\{0\}$, one certainly needs other methods. The core of the whole problem is to construct vector fields near f with the property that a trajectory of the induced (perturbed) dynamical system connects two points previously prescribed, say a_n and a_{n+1}. This can be done by standard methods of control and/or viability theory [1] as well but we did not succeed in finding estimates sharp enough. For discrete time, the extension problem for homeomorphisms can not be avoided by extending vector fields. Consequently, for discrete dynamical systems, our method breaks down and we wonder if the analogue of the Theorem is true or not. On the other hand, there is no difficulty if one considers discrete semidynamical systems on R^{p-1}. Proposition: Given a continuous (noninvertible) mapping $\varphi: R^{p-1} \to R^{p-1}$ and a chain-recurrent set $N \subset R^{p-1}$, then there is a continuous mapping $\mathscr{F}: R^{p-1} \times R_+ \to R^{p-1} \times R_+$ with the properties that $\mathscr{F}(x, 0) = (\varphi(x), 0)$ for all $x \in R^{p-1}$ and that $\omega((x_0, \lambda_0)) = \{(m, 0) \in R^{p-1} \times R_+ \mid m \in M\}$ for some $(x_0, \lambda_0) \in R^{p-1} \times (R_+ \setminus \{0\})$. The proof is easy: the "preparatory step" (see the proof of the Theorem, an adaptation of the arguments from [5, p. 334]) can immediately be followed by a simple application of Tietze extension theorem.

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